

**What Is Claimed Is:**

1        1.        A method for using a computer system to solve a system of  
2        nonlinear equations specified by a vector function,  $\mathbf{f}$ , wherein  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  represents a  
3        set of nonlinear equations,  $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , wherein  $\mathbf{x}$   
4        is a vector  $(x_1, x_2, x_3, \dots, x_n)$ , the method comprising:  
5                receiving a representation of a subbox  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , wherein for  
6                each dimension,  $i$ , the representation of  $X_i$  includes a first floating-point number,  
7                 $a_i$ , representing the left endpoint of  $X_i$ , and a second floating-point number,  $b_i$ ,  
8                representing the right endpoint of  $X_i$ ;  
9                storing the representation in a computer memory;  
10                applying term consistency to the set of nonlinear equations,  $f_1(\mathbf{x}) = 0,$   
11         $f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , over  $\mathbf{X}$ , and excluding portions of  $\mathbf{X}$  that violate  
12        any of these nonlinear equations;  
13                applying box consistency to the set of nonlinear equations over  $\mathbf{X}$ , and  
14                excluding portions of  $\mathbf{X}$  that violate any of the nonlinear equations; and  
15                performing an interval Newton step on  $\mathbf{X}$  to produce a resulting subbox  $\mathbf{Y}$ ,  
16        wherein the point of expansion of the interval Newton step is a point  $\mathbf{x}$  within  $\mathbf{X}$ ,  
17        and wherein performing the interval Newton step involves evaluating  $\mathbf{f}(\mathbf{x})$  using  
18        interval arithmetic to produce an interval result  $\mathbf{f}'(\mathbf{x})$ .

1        2.        The method of claim 1, wherein performing the interval Newton  
2        step involves:  
3                computing  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the function  $\mathbf{f}$   
4        evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ; and  
5                determining if  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular as a byproduct of solving for the subbox  $\mathbf{Y}$   
6        that contains values of  $\mathbf{y}$  that satisfy  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where

1       **M(x,X) = BJ(x,X)**, **r(x) = -Bf(x)**, and **B** is an approximate inverse of the center of  
2       **J(x,X)**.

1           3.       The method of claim 2, further comprising:  
2               applying term consistency to the preconditioned set of nonlinear equations  
3       **Bf(x) = 0** over the subbox **X**; and  
4               excluding portions of **X** that violate the preconditioned set of nonlinear  
5       equations.

1           4.       The method of claim 2, further comprising:  
2               applying box consistency to the preconditioned set of nonlinear equations  
3       **Bf(x) = 0** over the subbox **X**; and  
4               excluding portions of **X** that violate the preconditioned set of nonlinear  
5       equations.

1           5.       The method of claim 1, wherein applying term consistency to the  
2       set of nonlinear equations involves:  
3               for each nonlinear equation  $f_i(x) = 0$  in the system of equations  $\mathbf{f}(x) = \mathbf{0}$ ,  
4       symbolically manipulating  $f_i(x) = 0$  to solve for an invertible term,  $g(x'_j)$ , thereby  
5       producing a modified equation  $g(x'_j) = h(x)$ , wherein  $g(x'_j)$  can be analytically  
6       inverted to produce an inverse function  $g^{-1}(y)$ ;  
7               substituting the subbox **X** into the modified equation to produce the  
8       equation  $g(X'_j) = h(X)$ ;  
9               solving for  $X'_j = g^{-1}(h(X))$ ; and  
10          intersecting  $X'_j$  with the vector element  $X_j$  to produce a new subbox **X<sup>+</sup>**;

11           wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the system of  
12   equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  within the subbox  $\mathbf{X}$ , and wherein the width of the new subbox  
13    $\mathbf{X}^+$  is less than or equal to the width of the subbox  $\mathbf{X}$ .

1           6.       The method of claim 1, further comprising:  
2           evaluating a first termination condition, wherein the first termination  
3   condition is TRUE if,  
4                   zero is contained within  $\mathbf{f}^1(\mathbf{x})$ ,  
5                    $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular, wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the  
6   function  $\mathbf{f}$  evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ , and  
7                   the solution  $\mathbf{Y}$  of  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}$  contains  $\mathbf{X}$ ; and  
8                   if the first termination condition is TRUE, terminating and recording  $\mathbf{X}$  as  
9   a final bound.

1           7.       The method of claim 6, wherein the method further comprises:  
2           evaluating a second termination condition;  
3           wherein the second termination condition is TRUE if a function of the  
4   width of the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_X$ , and the width of the  
5   function  $\mathbf{f}$  over the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_F$ ; and  
6                   if the second termination condition is TRUE, terminating and recording  $\mathbf{X}$   
7   as a final bound.

1           8.       A computer-readable storage medium storing instructions that  
2   when executed by a computer cause the computer to perform a method for using a  
3   computer system to solve a system of nonlinear equations specified by a vector  
4   function,  $\mathbf{f}$ , wherein  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  represents a set of nonlinear equations,  $f_I(\mathbf{x}) = 0$ ,

5       $f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , wherein  $\mathbf{x}$  is a vector  $(x_1, x_2, x_3, \dots, x_n)$ , the  
6      method comprising:

7            receiving a representation of a subbox  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , wherein for  
8      each dimension,  $i$ , the representation of  $X_i$  includes a first floating-point number,  
9       $a_i$ , representing the left endpoint of  $X_i$ , and a second floating-point number,  $b_i$ ,  
10     representing the right endpoint of  $X_i$ ;

11            storing the representation in a computer memory;

12            applying term consistency to the set of nonlinear equations,  $f_1(\mathbf{x}) = 0,$   
13       $f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , over  $\mathbf{X}$ , and excluding portions of  $\mathbf{X}$  that violate  
14      any of these nonlinear equations;

15            applying box consistency to the set of nonlinear equations over  $\mathbf{X}$ , and  
16      excluding portions of  $\mathbf{X}$  that violate any of the nonlinear equations; and

17            performing an interval Newton step on  $\mathbf{X}$  to produce a resulting subbox  $\mathbf{Y}$ ,  
18      wherein the point of expansion of the interval Newton step is a point  $\mathbf{x}$  within  $\mathbf{X}$ ,  
19      and wherein performing the interval Newton step involves evaluating  $\mathbf{f}(\mathbf{x})$  using  
20      interval arithmetic to produce an interval result  $\mathbf{f}^1(\mathbf{x})$ .

1            9.      The computer-readable storage medium of claim 8, wherein  
2      performing the interval Newton step involves:

3            computing  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the function  $\mathbf{f}$   
4      evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ; and

5            determining if  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular as a byproduct of solving for the subbox  $\mathbf{Y}$   
6      that contains values of  $\mathbf{y}$  that satisfy  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where  
7       $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$ , and  $\mathbf{B}$  is an approximate inverse of the center of  
8       $\mathbf{J}(\mathbf{x}, \mathbf{X})$ .

1           10. The computer-readable storage medium of claim 9, wherein the  
2 method further comprises:

3           applying term consistency to the preconditioned set of nonlinear equations  
4  $\mathbf{Bf(x)} = \mathbf{0}$  over the subbox  $\mathbf{X}$ ; and  
5           excluding portions of  $\mathbf{X}$  that violate the preconditioned set of nonlinear  
6 equations.

1           11. The computer-readable storage medium of claim 9, wherein the  
2 method further comprises:

3           applying box consistency to the preconditioned set of nonlinear equations  
4  $\mathbf{Bf(x)} = \mathbf{0}$  over the subbox  $\mathbf{X}$ ; and  
5           excluding portions of  $\mathbf{X}$  that violate the preconditioned set of nonlinear  
6 equations.

1           12. The computer-readable storage medium of claim 8, wherein  
2 applying term consistency to the set of nonlinear equations involves:

3           for each nonlinear equation  $f_i(\mathbf{x}) = 0$  in the system of equations  $\mathbf{f(x)} = \mathbf{0}$ ,  
4 symbolically manipulating  $f_i(\mathbf{x}) = 0$  to solve for an invertible term,  $g(x'_j)$ , thereby  
5 producing a modified equation  $g(x'_j) = h(\mathbf{x})$ , wherein  $g(x'_j)$  can be analytically  
6 inverted to produce an inverse function  $g^{-1}(\mathbf{y})$ ;

7           substituting the subbox  $\mathbf{X}$  into the modified equation to produce the  
8 equation  $g(X'_j) = h(\mathbf{X})$ ;

9           solving for  $X'_j = g^{-1}(h(\mathbf{X}))$ ; and  
10           intersecting  $X'_j$  with the vector element  $X_j$  to produce a new subbox  $\mathbf{X}^+$ ;

11           wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the system of  
12 equations  $\mathbf{f(x)} = \mathbf{0}$  within the subbox  $\mathbf{X}$ , and wherein the width of the new subbox  
13  $\mathbf{X}^+$  is less than or equal to the width of the subbox  $\mathbf{X}$ .

1        13. The computer-readable storage medium of claim 8, wherein the  
2 method further comprises:

3                evaluating a first termination condition, wherein the first termination  
4 condition is TRUE if,

5                        zero is contained within  $\mathbf{f}^1(\mathbf{x})$ ,

6                         $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular, wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the  
7 function  $\mathbf{f}$  evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ , and

8                        the solution  $\mathbf{Y}$  of  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}$  contains  $\mathbf{X}$ ; and

9                        if the first termination condition is TRUE, terminating and recording  $\mathbf{X}$  as  
10 a final bound.

1        14. The computer-readable storage medium of claim 13, wherein the  
2 method further comprises:

3                evaluating a second termination condition;

4                wherein the second termination condition is TRUE if a function of the  
5 width of the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_X$ , and the width of the  
6 function  $\mathbf{f}$  over the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\varepsilon_F$ ; and

7                if the second termination condition is TRUE, terminating and recording  $\mathbf{X}$   
8 as a final bound.

1        15. An apparatus that solves a system of nonlinear equations specified  
2 by a vector function,  $\mathbf{f}$ , wherein  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  represents a set of nonlinear equations,  
3  $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , wherein  $\mathbf{x}$  is a vector  $(x_1, x_2, x_3, \dots, x_n)$ ,  
4 the apparatus comprising:

5                a receiving mechanism that is configured to receive a representation of a  
6 subbox  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ , wherein for each dimension,  $i$ , the representation of

7      $X_i$  includes a first floating-point number,  $a_i$ , representing the left endpoint of  $X_i$ ,  
8     and a second floating-point number,  $b_i$ , representing the right endpoint of  $X_i$ ;  
9                 a computer memory for storing the representation;  
10                a term consistency mechanism that is configured to apply term consistency  
11     to the set of nonlinear equations,  $f_1(\mathbf{x}) = 0, f_2(\mathbf{x}) = 0, f_3(\mathbf{x}) = 0, \dots, f_n(\mathbf{x}) = 0$ , over  
12      $\mathbf{X}$ , and to exclude portions of  $\mathbf{X}$  that violate any of these nonlinear equations;  
13                a box consistency mechanism that is configured to apply box consistency  
14     to the set of nonlinear equations over  $\mathbf{X}$ , and to exclude portions of  $\mathbf{X}$  that violate  
15     any of the nonlinear equations; and  
16                an interval Newton mechanism that is configured to perform an interval  
17     Newton step on  $\mathbf{X}$  to produce a resulting subbox  $\mathbf{Y}$ , wherein the point of  
18     expansion of the interval Newton step is a point  $\mathbf{x}$  within  $\mathbf{X}$ , and wherein  
19     performing the interval Newton step involves evaluating  $\mathbf{f}(\mathbf{x})$  using interval  
20     arithmetic to produce an interval result  $\mathbf{f}^I(\mathbf{x})$ .

1               16.     The apparatus of claim 15, wherein the interval Newton  
2     mechanism is configured to:  
3                compute  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the function  $\mathbf{f}$  evaluated  
4     as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ; and  
5                determine if  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular as a byproduct of solving for the subbox  $\mathbf{Y}$   
6     that contain the values of  $\mathbf{y}$  that satisfy  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where  
7      $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$ , and  $\mathbf{B}$  is an approximate inverse of the center of  
8      $\mathbf{J}(\mathbf{x}, \mathbf{X})$ .

1               17.     The apparatus of claim 16, wherein the term consistency  
2     mechanism is configured to:

1           apply term consistency to the preconditioned set of nonlinear equations  
2   **Bf(x) = 0** over the subbox **X**; and to  
3           exclude portions of **X** that violate the preconditioned set of nonlinear  
4   equations.

1           18.    The apparatus of claim 16, wherein the box consistency  
2  mechanism is configured to:  
3           apply box consistency to the preconditioned set of nonlinear equations  
4   **Bf(x) = 0** over the subbox **X**; and to  
5           exclude portions of **X** that violate the preconditioned set of nonlinear  
6  equations.

1           19.    The apparatus of claim 15, wherein for each nonlinear equation  
2  $f_i(\mathbf{x}) = 0$  in the system of equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ , the term consistency mechanism is  
3  configured to:  
4           symbolically manipulate  $f_i(\mathbf{x})=0$  to solve for an invertible term,  $g(x'_j)$ ,  
5  thereby producing a modified equation  $g(x'_j) = h(\mathbf{x})$ , wherein  $g(x'_j)$  can be  
6  analytically inverted to produce an inverse function  $g^{-1}(\mathbf{y})$ ;  
7           substitute the subbox **X** into the modified equation to produce the equation  
8  $g(X'_j) = h(\mathbf{X})$ ;  
9           solve for  $X'_j = g^{-1}(h(\mathbf{X}))$ ; and to  
10          intersect  $X'_j$  with the vector element  $X_j$  to produce a new subbox **X**<sup>+</sup>;  
11          wherein the new subbox **X**<sup>+</sup> contains all solutions of the system of  
12  equations  $\mathbf{f}(\mathbf{x}) = \mathbf{0}$  within the subbox **X**, and wherein the width of the new subbox  
13  **X**<sup>+</sup> is less than or equal to the width of the subbox **X**.

1           20. The apparatus of claim 15, further comprising a termination  
2 mechanism that is configured to:  
3           evaluate a first termination condition, wherein the first termination  
4 condition is TRUE if,  
5           zero is contained within  $\mathbf{f}^1(\mathbf{x})$ ,  
6            $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular, wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the  
7 function  $\mathbf{f}$  evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ , and  
8           the solution  $\mathbf{Y}$  of  $\mathbf{M}(\mathbf{x}, \mathbf{X}) (\mathbf{y} - \mathbf{x}) = \mathbf{r}$  contains  $\mathbf{X}$ ; and to  
9           terminate and record  $\mathbf{X}$  as a final bound if the first termination condition is  
10          TRUE.

1           21. The apparatus of claim 20, wherein the termination mechanism is  
2 additionally configured to:  
3           evaluate a second termination condition;  
4           wherein the second termination condition is TRUE if a function of the  
5 width of the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\epsilon_X$ , and the width of the  
6 function  $\mathbf{f}$  over the subbox  $\mathbf{X}$  is less than a pre-specified value,  $\epsilon_F$ ; and to  
7           terminate and record  $\mathbf{X}$  as a final bound if the second termination  
8 condition is TRUE.